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Completions of partial metrics into value lattices

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ABSTRACT

In this paper we investigate some notions of completion of partial metric spaces, including the bicompletion, the Smyth completion, and a new "spherical completion". Given an auxiliary relation, we show that it arises from a totally bounded partial metric space, and the spherical completion of such a space is its round ideal completion. We also give an example of a totally bounded partial metric space whose bicompletion and Smyth completion are not continuous posets. Finally, we present an example of a totally bounded partial metric giving rise to the Scott and lower topologies of a continuous poset, but whose spherical completion is not a continuous poset.

0. Introduction

Partial metrics were introduced in the early 1990s by the second author, to model computation over a metric space (X, d). In order to compute any given point $x \in X$ he proposed supplementing X with the set of *parts* of each of its elements, which would have to be computed in order to compute the element. It emerged that a generalized metric space, termed a *partial metric space* [10], could capture the structures of both the original metric space and the additional *partial points* P, by dropping the fundamental axiom of metric space theory that each self-distance $d(x,x)$ is necessarily zero. It is the case that for $x \in P \cup X$, $d(x,x) = 0$ if and only if $x \in X$, so the metric structure on X is preserved. Also, the partial points are nicely characterized as being exactly those points having nonzero self distance.

The authors later validated this proposal by proving that each *Scott domain* (the generally accepted structure of a *computable space*—see [1] for a discussion of this) is representable by a partial metric space, and in fact, so can each poset with auxiliary relation. But these do not always have a countable base, and when there is no such base, the partial metric must be valued not in the reals, but a power of the reals or of the unit interval. This work is in [9].

Key metric notions of topology, proximity and uniformity, generalize to the partial metric situation. This can be done strictly in terms of partial metrics (see [10]), but we choose instead to notice quasimetrics and metrics that arise from the partial metric, and we define topological and uniform notions in terms of these. M.B. Smyth pointed out (see [11]) that the bicompletion of a quasimetric space need not have suprema for its (specialization order-)directed sets, and thus might not be a continuous poset; he proposed an alternate completion, since known as the Smyth completion, in which each left Cauchy net has a τ_{q^s}-limit. Thus the bicompletion is a subspace of the Smyth completion, and in the latter, directed sets have suprema. We propose an alternative: In fact, as previously noted each poset with approximating auxiliary relation has its structure induced by a partial metric into a power of the unit interval. This latter space is totally bounded, and as a result it turns out that each directed net in the space is Cauchy (see 2.13(a)), and so it has a limit in the bicompletion, which must

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be the supremum of its range (see 2.4). Therefore, the bicompletion in this situation does have suprema of directed sets. It turns out that if a poset with approximating auxiliary relation has suprema for pairs that are bounded above, then both the bicompletion and the spherical completion of a totally bounded partial metric that induces the auxiliary relation, is its round ideal completion (see 3.4(b)). But whether or not such pairs have suprema, the spherical completion (defined in 2.1), in which only the increasing Cauchy nets are required to have limits, is always the round ideal completion (see 3.4(a)). Note that the spherical completion is always a subspace of the bicompletion, and is usually a proper subspace of it.

To avoid confusion, the reader should note an important alternate use of the word "complete" in the theory of computation. Points which can be computed in a metric space (X, d), are the result of a *completed* computation. W.W. Wadge, a mathematician and computer scientist, insightfully proposed: "A complete object (in a domain of data objects) is, roughly speaking, one which has no holes or gaps in it, one which cannot be further completed" [12]. Each part of x, being as it is a partial point, is a model of a partially completed computation. In the view of partial metric spaces, a partial point will be one whose self distance is nonzero. It is a partial characterization of what could become (after more computation) a point in X. In this view, the original metric space (X, d) is extended to a partial metric space (Y, p) which includes these partial points as points whose self distance is not zero.

The above notion is quite different from the established notion of a *complete metric space*, which has limits for all Cauchy sequences. But these ideas are related. Lawson conjectured, and the first author, in joint work, showed that a metrizable topological space is homeomorphic to the space of maximal (= complete) points of a Scott domain if and only if it arises from a complete metric (see [8]).

1. Value lattices and partial metrics

In this paper we use the standard conventions that for a poset $(P, \leqslant)$, if $A \subseteq P$, then $\uparrow A = \{q\colon \text{ for some } p \in A,\ p \leqslant q\}$ and for $p \in P$, $\uparrow p = \uparrow\{p\}$; also, A is an *upper set* if $A = \uparrow A$. Also $\downarrow$, $\Uparrow$, and $\Downarrow$ and *lower set* are similarly defined using $\geqslant$, $\ll$ and $\gg$, respectively. The *lower topology,* ω, (or $\omega_{(P,\leqslant)}$) of $(P, \leqslant)$, is the one whose closed sets are generated by $\{\uparrow p\colon\ p \in P\}$. Recall that a *directed complete poset,* or *dcpo* is a poset in which each directed set has a supremum. If $(P, \leqslant)$, is a dcpo, its *Scott topology,* σ (or $\sigma_{(P,\leqslant)}$) is the one whose closed sets are all lower sets C such that whenever D is a directed subset of C then $\bigvee D \in C$ and its *Lawson topology,* λ (or $\lambda_{(P,\leqslant)}$) is the join of its Scott and lower topologies. A dcpo is *continuous* if for each $p \in P$, $\Downarrow p$ is directed and $p = \bigvee \Downarrow p$, and a *continuous lattice* is a complete lattice which is a continuous dcpo. For in depth discussion of these concepts and their motivations, see [1] and [5].

Definition 1.1. A *value lattice* is a poset $(\mathcal{V}, \leqslant)$, whose least element is denoted 0 and largest is ∞, such that $(\mathcal{V}, \geqslant)$ is a continuous lattice, together with an associative, commutative operation $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ such that 0 is an identity and for each $R, S \subseteq \mathcal{V}$, $(\bigwedge R) + (\bigwedge S) = \bigwedge\{r + s\colon\ r \in R,\ s \in S\}$, where $\bigwedge$ denotes inf.

Here are some simple but useful consequences of this infinite distributive law:

(s1) For all $p \in V$, $p + \infty = \infty$.
(s2) For all $p, q, r, s \in V$, $p \geqslant q$ and $r \geqslant s$ implies $p + r \geqslant q + s$.

For value lattices, our notation and definition is as in [2] and [9], except that we feature the bottom and top elements in our notation: $\mathcal{V} = \langle V, \leqslant, +, 0, 1\rangle$. The bottom will always be denoted 0, the top usually 1, unless general practice suggests ∞. Note that these are simply generalizations of the extended non-negative reals, $\mathbb{E} = [0, \infty]$, with the usual order, addition and constants; other key examples are the unit interval, $\mathbb{I} = [0, 1]$, and the Boolean set, $\mathbb{B} = \{0, \infty\}$, with the expected order and operations, except that in $\mathbb{I}$, addition is truncated: $a + b = \min\{a +_u b, 1\}$, where $+_u$ denotes the usual addition. For each value lattice there is a "subtraction", $\dot{-}$, defined to be the left adjoint of addition: $p \dot{-} q = \bigwedge\{r \mid p \leqslant q + r\}$. Then $p \dot{-} q \leqslant r \Leftrightarrow p \leqslant q + r$, a fact regularly used below. Also we define $|x \dot{-} y| = (x \dot{-} y) \vee (y \dot{-} x)$.

Value lattices and quantales are discussed in [9], and below. Products of value lattices are value lattices, and our distance functions will always go into a power of $\mathbb{I}$, $\mathbb{E}$, or $\mathbb{B}$; indeed our counterexamples use real valued partial metrics.

We are particularly interested in two types of distance functions into value lattices:

Definition 1.2. A $\mathcal{V}$-quasimetric (or $\mathcal{V}$-qmetric) is a function $q : X \times X \to \mathcal{V}$, satisfying, for every $x, y, z \in X$:

(Qid) $q(x, x) = 0$,
(Qtr) $q(x, z) \leqslant q(x, y) + q(y, z)$,
(Qt0) if $q(x, y) + q(y, x) = 0$, then $x = y$.

A set X with a $\mathcal{V}$-qmetric $q : X \times X \to \mathcal{V}$, is then called a $\mathcal{V}$-qmetric space.

A *$\mathcal{V}$-partial metric* ($\mathcal{V}$-pmetric) is a function $p : X \times X \to \mathcal{V}$ satisfying the following weakening of identity, and other metric conditions, including symmetry. For every $x, y, z \in X$:

(Pid) $p(x, y) \geqslant p(x, x)$,
(Psy) $p(x, y) = p(y, x)$,

(Ptr) $p(x,z) \leqslant p(x,y) + (p(y,z) \dot{-} p(y,y))$,
(Pt0) $x = y$ if $p(x,y) = p(x,x) = p(y,y)$.

A $\mathcal{V}$-pmetric space is a set X and a $\mathcal{V}$-pmetric $p : X \times X \to \mathcal{V}$. We often omit $\mathcal{V}$- in these notations, when it is clear from the context.

The *dual* of any qmetric is the qmetric defined by $q^*(x,y) = q(y,x)$ and its *symmetrization* is the qmetric $q^s = q + q^*$. Clearly, $q^s(x,y) = q^s(y,x)$, so q^s is a $\mathcal{V}$-metric in the (Qt0) case.

Given a $\mathcal{V}$-pmetric, its *associated* $\mathcal{V}$*-qmetric* is $q_p : X \times X \to \mathcal{V}$, defined by $q_p(x,y) = p(x,y) \dot{-} p(x,x)$, so $q_p^*(x,y) = p(x,y) \dot{-} p(y,y)$ and $q_p^s(x,y) = (p(x,y) \dot{-} p(x,x)) + (p(x,y) \dot{-} p(y,y))$. If $p : X \times X \to \mathcal{V}$ is a pmetric then q_p is a qmetric, as shown in [9].

Note that (Pt0) says that if $q_p(x,y) + q_p(y,x) = 0$ then $x = y$; that is, (Qt0) for the qmetric q_p. Different papers give different definitions of quasimetrics; in particular, quasimetrics in [2] and [9] were not required to satisfy (Qt0). We use a different (though common) definition here because we want:

- our partial metrics and quasimetrics to give rise to T_0 topologies (those for which $x \in \mathrm{cl}(\{y\}) \& y \in \mathrm{cl}(\{x\}) \Rightarrow x = y$), and
- their associated pseudometrics to be metrics, so that their topologies are Hausdorff and their limits unique.

Definition 1.3. Given a $\mathcal{V}$-qmetric $q : X \times X \to \mathcal{V}$, the *closed ball* about $x \in X$ of radius $r \in \mathcal{V}$ is the set $N_r(x) = \{y \in X\colon\ q(x,y) \leqslant r\}$; also, $N_r^*(x) = \{y \in X\colon\ q^*(x,y) \leqslant r\}$. The *open ball* about x of radius r is $B_r(x) = \{y\colon\ q(x,y) \ll r\}$, where $\ll$ refers to the "way-below" relation on $(\mathcal{V}, \geqslant)$, defined by $r \ll s \Leftrightarrow$ whenever $r \geqslant \bigwedge D$, $(D, \geqslant)$ directed, then for some $d \in D$, $s \geqslant d$.

In [2] and [9] it is shown that for each $\mathcal{V}$-qmetric, τ_q is a topology, where a subset U of X is open in τ_q if for each $x \in U$ there is an $r \gg 0$ such that $N_r(x) \subseteq U$. Then τ_q is called the *topology induced by* q. For each $x \in X$, the set of open balls $\{B_r(x)\colon\ r \gg 0\}$ and the set of closed balls $\{N_r(x)\colon\ r \gg 0\}$, are neighborhood bases for τ_q at x. Also, for each $x \in X$, $r \in \mathcal{V}$, $N_r(x)$ is a closed set in τ_{q^*}, and if $r \gg 0$ then $B_r(x)$ is an open set in τ_q. In particular, the set of open balls with arbitrary centers, is a base for the topology τ_q.

Uniform notions can also be defined in terms of a $\mathcal{V}$-qmetric $q : X \times X \to \mathcal{V}$: the *entourage of radius* $r \in \mathcal{V}$ is the set $N_r = \{(x,y) \in X \times X\colon\ q(x,y) \leqslant r\}$, and the *quasiuniformity induced by* q, $\mathcal{Q}_q$, is the set $\{U \subseteq X \times X\colon$ for some $r \gg 0,\ N_r \subseteq U\}$, which is easily shown to be a quasiuniformity. A clear, comprehensive discussion of quasiuniformities is found in [4].

Topological notions also result from $\mathcal{V}$-pmetrics, and these are all defined in terms of the associated $\mathcal{V}$-qmetric., e.g.: $\tau_p = \tau_{q_p}$, $\tau_{p^*} = \tau_{(q_p)^*}$, and $\mathcal{Q}_p = \mathcal{Q}_{q_p}$.

In fact it is proved below that for each value lattice $\mathcal{V}$, $p_{\mathcal{V}}(x,y) = x \vee y$ is a $\mathcal{V}$-pmetric on it that gives rise to its induced topologies, and which we call its *natural* $\mathcal{V}$-pmetric:

Theorem 1.4. *Given a value lattice $\mathcal{V}$, define $p_{\mathcal{V}} : \mathcal{V} \times V \to V$ by $p_{\mathcal{V}}(x,y) = x \vee y$. Then $p_{\mathcal{V}}$ is a $\mathcal{V}$-pmetric. Further, $\tau_{p_{\mathcal{V}}} = \sigma_{(\mathcal{V},\geqslant)}$ and $\tau_{p^*_{\mathcal{V}}} = \omega_{(\mathcal{V},\geqslant)}$.*

Proof. Axioms (Pid) and (Psy) are clear; for (Ptr), notice first that on $\mathcal{V}$, $u \leqslant (u \dot{-} v) + v$ simply because $u \dot{-} v \leqslant (u \dot{-} v)$. Since $v \leqslant v + (u \dot{-} v)$ also, we have that $u \vee v \leqslant v + (u \dot{-} v)$.

Using the facts shown in the last paragraph on $u = z$, $v = x \vee y$, and the monotonicity of $\dot{-}$ (if $s \leqslant s'$ and $t \geqslant t'$ then $s \dot{-} t \leqslant s' \dot{-} t'$), we have that $x \vee y \vee z \leqslant x \vee y + (z \dot{-} x \vee y) \leqslant x \vee y + (z \dot{-} y) \leqslant x \vee y + ((y \vee z) \dot{-} y)$. Thus $p_{\mathcal{V}}(x,z) = x \vee z \leqslant x \vee y \vee z \leqslant (x \vee y) + ((y \vee z) \dot{-} y) = p_{\mathcal{V}}(x,y) + (p_{\mathcal{V}}(y,z) \dot{-} p_{\mathcal{V}}(y,y))$.

For (Pt0), simply note that $x \vee y = x \vee x = y \vee y$ certainly implies $x = y$.

From the inequality $u \vee v \leqslant v + (u \dot{-} v)$ at the end of the first paragraph of the proof above, we also get $(v \vee u) \dot{-} v \leqslant u \dot{-} v$, thus $(v \vee u) \dot{-} v = u \dot{-} v$ since $\geqslant$ is clear. So letting $q_{\mathcal{V}}(x,y) = p_{\mathcal{V}}(x,y) \dot{-} p_{\mathcal{V}}(x,x) = x \vee y \dot{-} x = y \dot{-} x$, we also get $q^*_{\mathcal{V}}(x,y) = x \dot{-} y$; and $q^s_{\mathcal{V}}(x,y) = (x \dot{-} y) + (y \dot{-} x) \leqslant 2|x \dot{-} y|$.

In [2, 4.9] (also see [9]), it was shown that for each value lattice $\mathcal{V}$, $\tau_{q_{\mathcal{V}}}$ is the Scott topology for the continuous dcpo order, $\geqslant$, and since $q_{\mathcal{V}} = q_{p_{\mathcal{V}}}$, our results follow. $\square$

Note that for ϵ, $a, x \in \mathcal{V}$, $x \dot{-} a \leqslant \epsilon \Leftrightarrow x \leqslant a + \epsilon$, $a \dot{-} x \leqslant \epsilon \Leftrightarrow x \geqslant a \dot{-} \epsilon$, and so $|x \dot{-} a| \leqslant \epsilon \Leftrightarrow a \dot{-} \epsilon \leqslant x \leqslant a + \epsilon$. In particular, $|x \dot{-} a| = 0$ iff $x = a$.

A particular case of the above shows that $p_{\mathbb{R}}(x,y) = \max\{x,y\} : [0,\infty) \times [0,\infty) \to [0,\infty]$ is a partial metric. It is easy to check that $q_{\mathbb{R}}(x,y) = \max\{y - x, 0\}$, which gives rise to the lower topology on $\mathbb{R}$, whose nontrivial open sets are those of the form $(-\infty, a)$, $a \in \mathbb{R}$. Its dual gives rise to the Scott topology $\sigma_{(\mathbb{R},\leqslant)}$ on $\mathbb{R}$, and $q^s_{\mathbb{R}}(x,y) = |x - y|$, the usual real metric, giving the Euclidian topology on $\mathbb{R}$ (which is its Lawson topology).

We next show the uniform continuity of several often-used operations; for this, note that given two $\mathcal{V}$-qmetric spaces (X,q), (X',q'), the product $\mathcal{V}$-qmetric space is $X \times X'$, equipped with the following function, which is easily seen to be a $\mathcal{V}$-qmetric:

$$(q \times q')\big((x,y),(u,v)\big) = q(x,u) \vee q'(y,v).$$

Uniform and topological notions are easily characterized in terms of $\mathcal{V}$-qmetrics. For example, given $\mathcal{V}$-qmetric spaces (X,q), (X',q'):

A function $f : X \to X'$ is uniformly continuous from $(X, \mathcal{Q}_q)$ to $(X', \mathcal{Q}_{q'})$ if for each $r \gg 0$, there is an $s \gg 0$ such that $q(x, y) \leqslant s \Rightarrow q'(f(x), f(y)) \leqslant r$, and is continuous from $(X, \mathcal{Q}_q)$ to $(X', \mathcal{Q}_{q'})$ at $x \in X$, if for each $r \gg 0$ there is an $s \gg 0$ such that $q(x, y) \leqslant s \Rightarrow q'(f(x), f(y)) \leqslant r$. Clearly if $f : X \to X'$ is uniformly continuous from $(X, \mathcal{Q}_q)$ to $(X', \mathcal{Q}_{q'})$, then f is uniformly continuous from $(X, \mathcal{Q}_{q^*})$ to $(X', \mathcal{Q}_{q'^*})$ and from $(X, \mathcal{Q}_{q^s})$ to $(X', \mathcal{Q}_{q'^s})$.

Because of the above straightforward characterizations, we often say that functions are continuous, or uniformly continuous, with respect to q, q' or p when we mean that they are continuous with respect to the induced topologies or uniformly continuous with respect to the induced quasiuniformities. Similar conventions apply to nets.

Theorem 1.5.

(a) *If (X,q) is a $\mathcal{V}$-quasimetric space, then $q : (X,q) \times (X,q^*) \to (V, q^*_{\mathcal{V}})$ is uniformly continuous; as a special case, $\dot{-} : (\mathcal{V}, q^*_{\mathcal{V}}) \times (\mathcal{V}, q_{\mathcal{V}}) \to (\mathcal{V}, q_{\mathcal{V}})$ is uniformly continuous. Further, $+, \wedge, \vee : (\mathcal{V}, q_{\mathcal{V}}) \times (\mathcal{V}, q_{\mathcal{V}}) \to (\mathcal{V}, q_{\mathcal{V}})$ are uniformly continuous, and $\leqslant$ is $q^s_{\mathcal{V}}$-closed in $\mathcal{V} \times V$.*

(b) *If (X, p) is a $\mathcal{V}$-pmetric space, then p is uniformly continuous from $(X, q_p)^2$ to $(V, q_{\mathcal{V}})$, from $(X, q^*_p)^2$ to $(V, q^*_{\mathcal{V}})$, and from $(X, q^s_p)^2$ to $(V, q^s_{\mathcal{V}})$.*

Proof. (a) Two uses of the triangle inequality and the definition of $q \times q'$, give $q(u, v) \leqslant q(u,x) + q(x, y) + q(y, v) \leqslant q(x, y) + 2(q^* \times q)((x, y), (u, v))$. Therefore, $q^*_{\mathcal{V}}(q(u, v), q(x, y)) = q(u, v) \dot{-} q(x, y) \leqslant 2(q^* \times q)((u, v), (x, y))$ showing that q is uniformly continuous. The assertion on $\dot{-}$ follows, since $x \dot{-} y = q^*_{\mathcal{V}}(x, y)$.

The other assertions arise from the following inequalities, all of which result from the inequality $x \dot{-} y \leqslant z \Leftrightarrow x \leqslant y + z$:

$$(x + y) \dot{-} (u + v) \leqslant (x \dot{-} u) + (y \dot{-} v) \leqslant 2(q_{\mathcal{V}} \times q_{\mathcal{V}})\big((x, y), (u, v)\big),$$
$$(x \vee y) \dot{-} (u \vee v) \leqslant (x \dot{-} u) + (y \dot{-} v),$$
$$(x \wedge y) \dot{-} (u \wedge v) \leqslant (x \dot{-} u) + (y \dot{-} v).$$

(b) Two uses of the triangle inequality and symmetry give $p(x, y) \leqslant p(x, v) + (p(v, y) \dot{-} p(v, v)) \leqslant p(u, v) + (p(u,x) \dot{-} p(u,u)) + (p(y, v) \dot{-} p(v, v)) \leqslant p(u, v) + 2(q_p \times q_p)((x, y), (u, v))$. Thus $q_{\mathcal{V}}(p(u, v), p(x, y)) = p(x, y) \dot{-} p(u, v) \leqslant 2(q_p \times q_p)((u, v), (x, y))$, establishing uniform continuity from $(X, q_p)^2$ to $(V, q_{\mathcal{V}})$. But uniformly continuous functions from a given quasimetric to another, are uniformly continuous also with respect to their duals and their symmetrizations. That $\leqslant$ is closed results from the above, since if $x = \lim x_n$, $y = \lim y_n$, and each $x_n \leqslant y_n$, then $y = \lim(x_n \vee y_n) = x \vee y$, so $x \leqslant y$. □

2. Completions

We recall that given a $\mathcal{V}$-qmetric space (X,q), a net $x : (D, \leqslant) \to X$ is Cauchy if for each $r \gg 0$, there is an $n \in D$ such that $m, p \geqslant n \Rightarrow q(x_m, x_p) \leqslant r$. It turns out that given a $\mathcal{V}$-pmetric on a set X, a net $\langle x_n\rangle_{n\in D} \subseteq X$ is Cauchy with respect to q_p if and only if $\lim^s p(x_m, x_n)$ exists. To be a bit more explicit, when a net is Cauchy this means that there is unique $a \in \mathcal{V}$ such that: $\forall \epsilon \gg 0$ there is an M such that if $m,n \geqslant M$ then $(p(x_m, x_n) \dot{-} a) \vee (a \dot{-} p(x_m, x_n)) \leqslant \epsilon$.

Each T_0 topology induces a partial order, its *specialization order* given by $x \leqslant_\tau y \Leftrightarrow x \in \mathrm{cl}(\{y\})$. This order can be characterized in terms of any quasimetric or partial metric that induces the topology: $x \leqslant_{\tau_q} y \Leftrightarrow q(x, y) = 0$; $x \leqslant_{\tau_p} y \Leftrightarrow p(x, y) = p(x,x)$. Thus we use $\leqslant_q$ to denote $\leqslant_{\tau_q}$ and $\leqslant_p$ to denote $\leqslant_{\tau_p}$.

Definition 2.1. A qmetric space is *bicomplete* if every Cauchy net ^{s}-converges. A net is *spherically Cauchy* if it is Cauchy and $\leqslant_q$-increasing, and a qmetric space is *spherically complete* if every spherically Cauchy net ^{s}-converges. A *bicompletion* (resp. *spherical completion*) of (X,q), is a bicomplete (resp. spherically complete) qmetric space (Y,q') such that $Y \supseteq X$, $q = q'|X \times X$, and (Y,q') has no bicomplete (resp. spherically complete) proper subspaces that contain X.

It is clear by this definition that bicomplete $\mathcal{V}$-qmetric spaces are spherically complete. Also, for metric spaces, completeness is bicompleteness.

Since $\tau^s_{\mathcal{V}}$ is Hausdorff, if $\lim^s p(x_m, x_n)$ exists then it is unique. The following examples show various possibilities that can hold for spherical completeness and bicompleteness:

Examples 2.2. (a) Let $J = ([0, 1], p_J)$, where

$$p_J(x, y) = \begin{cases} 0, & x = y = 1, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Clearly p_J satisfies (Pid), (Psy), and (Pt0). For (Ptr), note that unless $y = 1$, by Theorem 1.4:

$$p_J(x,z) \leqslant x \vee z \leqslant x \vee y + (y \vee z - y \vee y) = p_J(x,y) + \big(p_J(y,z) \dot{-} p_J(y,y)\big).$$

But if $y = 1$ then these equations and inequalities hold except that the last $=$ becomes $\leqslant$, so (Ptr) holds anyway.

Now let $\langle x_i \rangle$ be a strictly increasing sequence approaching 1 in the Euclidean topology; then $\lim p(x_n, x_m) = 1 = \bigvee p(x_n, x_m)$. But $\{1\} = B_{.5}(1)$ thus is ^{s}-open. Thus x_n has no ^{s}-limit in X, so this sequence is spherically Cauchy but is not convergent. Therefore this space is not spherically complete, and thus not bicomplete.

(b) Interestingly, every metric space (X,d) is spherically complete since $\leqslant_d$ is equality, so each spherically Cauchy net is constant. Since completeness is bicompleteness for metrics, we have a large supply of spherically complete non-bicomplete spaces here.

(c) Consider any complete metric space (X,d) and let $p = d$. Then (X,p) is bicomplete, and since $\leqslant_p$ is equality, $(X, \leqslant_p)$ is trivially a dcpo, but is not a lattice.

Lemma 2.3.

(a) *In a $\mathcal{V}$-qmetric space q, each Cauchy net is Cauchy in both q^* and q^s. Also if $\lim^s x_n, \lim^s y_m$ exist, then $\lim^s q(x_n, y_m) = q(\lim^s x_n, \lim^s y_m)$.*
(b) *A net, $\langle x_i \rangle_{i \in I} \subseteq X$, in a $\mathcal{V}$-pmetric space is Cauchy with respect to q_p if and only if $\lim^s p(x_m, x_n)$ exists.*
(c) *Every ^{s}-convergent net in a $\mathcal{V}$-qmetric space is Cauchy.*

Proof. For (a), let the net $\langle x_i \rangle$ be Cauchy. For each $r \gg 0$, find $s \gg 0$ so that $2s \leqslant r$. Next find $n \in D$ such that $m, p \geqslant n \Rightarrow q(x_m, x_p) \leqslant s$. Then if $m, p \geqslant n$ we have $q^*(x_m, x_p) = q(x_p, x_m) \leqslant s$ and $q^s(x_m, x_p) = q(x_m, x_p) + q(x_p, x_m) \leqslant 2s \leqslant r$. The proofs of the rest of (a), and of (b) and (c), are left to the reader. □

Recall that a *bitopological space* is a space with an indexed set of two topologies, (X, τ, τ^*). Its *symmetrization topology* is $\tau^s = \tau \vee \tau^*$, and its *bitopological dual* is (X, τ^*, τ). Such a space is T_0 if τ^s is a T_0 topology, and ^{s}-compact if τ^s is a compact topology. It is T_1 if T_0 and $\leqslant_{\tau^*} \subseteq (\leqslant_\tau)^{-1}$. A map between two such spaces X, Y is *pairwise continuous* if whenever $T \in \tau_Y$ then $f^{-1}[T] \in \tau_X$ and whenever $T \in \tau_Y^*$ then $f^{-1}[T] \in \tau_X^*$, and such a space satisfies some property Q, *pairwise* if both it and its bitopological dual satisfy Q. See [7] for more on these spaces. The following result tells us very generally, that ^{s}-cluster points of directed sets are their suprema.

Lemma 2.4. *Let (X, τ, τ^*) be a pairwise T_1 bitopological space, and let $y : D \to X$ be an increasing net in X. Then the only possible cluster point of y in τ^s is $\bigvee y[D]$.*

Proof. Of course, if a net is eventually in some closed set, then any cluster point is contained there. Let x be an ^{s}-cluster point of y. If y is increasing for $\leqslant_\tau$ then given $d \in D$, if $d \leqslant e \in D$ and u is any upper bound for $y[D]$, we have $y_e \in (\uparrow y_d) \cap (\downarrow u) = \mathrm{cl}^*(\{y_d\}) \cap \mathrm{cl}(\{u\})$, an ^{s}-closed set, so the ^{s}-cluster point x is in $(\uparrow y_d) \cap (\downarrow u)$. This shows that $x = \bigvee y[D]$. □

Given any spherically complete $\mathcal{V}$-pmetric space, (X,p), if $D \subseteq X$ is $\leqslant_p$-directed, and the increasing net $1_D : D \to X$ is Cauchy, then it has an ^{s}-limit, which is an ^{s}-cluster point, so it must be $\bigvee 1_D[D] = \bigvee D$. Below, we often refer to limits of directed subsets of a poset X, with a topology. In these cases, we always have in mind for $D \subseteq X$, the limit of the identity map 1_D.

Theorem 2.5. *Every $\mathcal{V}$-pmetric space X, has a completion, $M(X)$ and a spherical completion $S(X)$.*

Further, if Z is a completion (resp. spherical completion) of X, then for each uniformly continuous $f : X \to W$, W a complete (resp. spherically complete) space, there is a unique $\hat{f} : Z \to W$ which is uniformly continuous, such that $f = \hat{f}|X$. As a result, the completion and the spherical completion of X are unique up to uniform isomorphism.

Proof. First note that q_p^s is a $\mathcal{V}$-metric. To see the four metric properties:

$$q_p^s(x,x) = q_p(x,x) + q_p^*(x,x) = 0,$$

$$q_p^s(x,z) = q_p(x,z) + q_p^*(x,z) \leqslant q_p(x,y) + q_p(y,z) + q_p^*(x,y) + q_p^*(y,z) = q_p^s(x,y) + q_p^s(y,z),$$

$$q_p^s(x,y) = q_p(x,y) + q_p(y,x) = q_p^s(y,x),$$

and the fact that if $q_p^s(x,y) = 0$ then $q_p(x,y) + q_p(y,x) = 0$, so by the paragraph preceding 1.3, $x = y$.

As a result, the usual proof that every metric space has a completion shows that (X, q_p^s) has a completion, which we call $(M(X), \hat{q}_p^s)$. Also, given $\mathcal{V}$-qmetric spaces (W, q_W), (Y, q_Y), any uniformly continuous $f : V \to Y$, $V \subseteq W$ is easily seen to be uniformly continuous between their symmetrizations, so if Y is complete and V is ^{s}-dense in W, then, by the usual proof, f extends uniquely to a uniformly continuous $\hat{f} : (W, q_W) \to (Y, q_Y)$. By Theorem 1.5(b), p is uniformly continuous from $(X, q_p^s)^2$ to $(\mathcal{V}, q_{\mathcal{V}}^s)$; so it extends uniquely to $\hat{p} : M(X) \times M(X) \to \mathcal{V}$, which is easily be seen to be a partial metric. For example, to see (Ptr), if $x, y, z \in M(X)$ then there are nets, which we may assume come from a single directed set D, such that $x_n \to x$, $y_n \to y$, and $z_n \to z$. By Theorem 1.5(a), $+, \dot{-} : \mathcal{V} \times V \to V$ are uniformly continuous and $\leqslant$ is closed in $\mathcal{V} \times \mathcal{V}$, so $\hat{p}(x, z) = \lim^s p(x_n, z_n) \leqslant \lim^s[p(x_n, y_n) + (p(y_n, z_n) \dot{-} p(y_n, y_n))] = \lim^s p(x_n, y_n) + (\lim^s p(y_n, z_n) \dot{-} \lim^s p(y_n, y_n)) = \hat{p}(x, y) + (\hat{p}(x, z) \dot{-} \hat{p}(y, y))$.

Since they are ^{s}-uniformly continuous and equal on the dense subspace $X \times X$, $q_{\hat{p}}^s = \hat{q}_p^s$. Thus, since $(M(X), \hat{q}_p)$ is a bicomplete quasimetric space, $(M(X), \hat{p})$ is a bicomplete partial metric space. No subspace is bicomplete since each point in $M(X)$ is the unique limit of a net in X, thus by the above, $M(X)$ is a bicompletion of X.

For the spherical completion, let $S(X)$ be the intersection of all spherically complete subspaces of $M(X)$ that contain X. We show that $S(X)$ is also spherically complete (it certainly contains X): for if $D \subseteq S(X)$ is Cauchy and directed, then for each spherically complete subspace Y of $M(X)$, $D \subseteq Y$, so $\bigvee D = \lim^s D \in Y$, thus $\bigvee D \in S(X)$, the intersection of all such Y. Also, no proper subspace of $S(X)$ contains X and is spherically complete, for if Z were such a subspace, it would be among those intersected to get $S(X)$, so $S(X) \subseteq Z$. By these properties, $S(X)$ is a spherical completion of X.

It remains to be seen that each uniformly continuous function $f : X \to W$, W spherically complete, extends uniquely to $S(X)$. By the previous discussion, f extends uniquely to a uniformly continuous $\vec{f} : M(X) \to M(W)$. As usual, f preserves Cauchy nets, and f is continuous so it preserves the specialization order. Thus if $\langle x_n \rangle$ is a directed Cauchy net in $S(X)$ then so is $\langle f(x_n) \rangle$, thus if each $f(x_n) \in W$, then $\lim f(x_n)$ exists in W. The above shows that $f^{-1}[W]$ contains X and is closed under limits of directed Cauchy nets, so $S(X) \subseteq f^{-1}[W]$. Therefore, $\hat{f} = \vec{f}|S(X) : S(X) \to W$, is a uniformly continuous extension of f to $S(X)$. It is unique because for any continuous extension g to $S(X)$, $g|X = f = \hat{f}|X$, so since $(M(x), \tau_{\hat{q}_p^s})$ is Hausdorff and X is dense, $g = \hat{f}$. □

The quasiuniform space $(M(X), \mathcal{Q}_{\hat{q}_p})$ is the bicompletion of the quasiuniform space $(X, \mathcal{Q}_{q_p})$ by Theorem 2.5 since the bicompletion of the latter is characterized by the fact that uniformly continuous maps from $(X, \mathcal{Q}_{q_p})$ to bicomplete spaces extend uniquely to it.

Definition 2.6. If $p : X \times X \to \mathcal{V}$ is a partial metric then for each $x \in X$ we define the function $p(x, -) : X \to \mathcal{V}$ by $p(x, -)(y) = p(x, y)$.

Theorem 2.7. *Let p be a partial metric on the set X to the value lattice $\mathcal{V}$. Then for every $x \in P$, the map $p(x, -) : (X, \leqslant_p) \to (\mathcal{V}, \geqslant_{\mathcal{V}})$ is order preserving.*

Proof. This is obvious since each continuous map preserves the specialization order. For a direct proof, consider $a, b \in X$ such that $b \leqslant_p a$. Then $p(x, a) \leqslant p(x, b) + (p(a, b) \dot{-} p(b, b)) = p(x, b) + 0 = p(x, b)$. Thus $p(x, -)$ is order preserving. □

The following gives a condition in which suprema are well behaved:

Definition 2.8. Let $\mathcal{V}$ be a value lattice and $p : X \times X \to \mathcal{V}$ be a $\mathcal{V}$-pmetric space. We say that p is *join preserving* if for every directed subset A of $(X, \leqslant_p)$ with a sup, and each $x \in X$, $p(x, \bigvee A) = \bigwedge_{a \in A} p(x, a)$.

It might seem perverse that join preserving is defined by the equation $p(x, \bigvee A) = \bigwedge_{a \in A} p(x, a)$. But in fact, the join on the left is with respect to $\leqslant_p$, and on the right, with respect to $\leqslant_{p_{\mathcal{V}}}$, which is $\geqslant$ on $\mathcal{V}$. The idea is that each $p(x, -)$ should be Scott continuous, as shown in the next theorem.

Examples 2.9. Each $\mathcal{V}$-metric space (X, d) is trivially spherically complete and a dcpo, and its metric is join preserving. All these result from the fact that $\leqslant_d$ is equality, so increasing nets are constant, and directed sets are singletons.

Theorem 2.10. *Let p be a partial metric on a poset $(P, \leqslant)$ to a value lattice $\mathcal{V}$. Then p is join preserving if and only if any of the following equivalent conditions holds*:

(a) $\tau_p \subseteq \sigma_{(P, \leqslant)}$,
(b) *each $p(x, -)$ is Scott continuous from $(P, \leqslant)$ to $(\mathcal{V}, \geqslant)$,*
(c) *$\leqslant \subseteq \leqslant_{\tau_p}$, and whenever D is directed with a join, $\bigvee D$ is a τ_p-limit of D,*
(d) *$\leqslant \subseteq \leqslant_{\tau_p}$, and whenever D is directed and $\bigvee D$ exists then $\bigvee D = \lim^s D$.*

Proof. For (a) ⇒ (b): by Theorem 1.5(b), $p : (P, q_p)^2 \to (V, q_{\mathcal{V}})$ is uniformly continuous; thus p is continuous from $\tau_{q_p}^2$ to $\sigma_{(\mathcal{V}, \geqslant)}$, so for each $x \in P$, $p(x, -)$ is continuous from $\sigma_{(P, \leqslant)}$ to $\sigma_{(\mathcal{V}, \geqslant)}$. For (b) ⇒ (a), if for each x, the map $p(x, -)$ takes directed suprema to infima, it is continuous from $(P, \sigma_{(P, \leqslant)})$ to $(\mathcal{V}, \sigma_{\geqslant}) = (V, \tau_{p_{\mathcal{V}}})$, so each $B_\epsilon(x) = p(x, -)^{-1}[\Downarrow(p(x, x) + \epsilon)]$ must be σ-open, so $\tau_p \subseteq \sigma_{(P, \leqslant)}$.

To see (c) $\Rightarrow$ (d), since for each directed D, $\bigvee D$ is a limit for D in $\sigma = \tau_p$, it suffices to note that it is also a τ_{p^*}-limit of D; but $\leqslant \subseteq \leqslant_{\tau_p} = \geqslant_{\tau_{p^*}}$, so each τ^*-open set containing $\bigvee D$ must contain D; the reverse, (d) $\Rightarrow$ (c), is immediate since all τ_{p^s}-limits are τ_p-limits.

To see that (c) $\Rightarrow$ (a), simply note that since $\leqslant \subseteq \leqslant_{\tau_p}$, each τ_p-closed set C is a lower set, and since for each directed $D \subseteq C$, $\bigvee D$ is a limit of D, we have that $\bigvee D \in C$, so C is Scott closed; this shows that $\tau_p \subseteq \sigma_{(P,\leqslant)}$. Finally, to see that (a) $\Rightarrow$ (c), if $\tau_p \subseteq \sigma_{(P,\leqslant)}$, then $\leqslant_{\tau_p} \supseteq \leqslant_{\sigma_{(P,\leqslant)}} = \leqslant$, and whenever D is directed with a join, $\bigvee D$ is a $\sigma_{(P,\leqslant)}$-limit of D, so it is a τ_p-limit of D. $\square$

Example 2.11. An example of a space whose pmetric is not join preserving is $J = [0, s) \cup \{1\}$, $s < 1$ together with $p(x, y) = x \vee y$. Then $1 = \bigvee[0, s)$ but $p(0, \bigvee[0, s)) = p(0, 1) = 1 \neq s = \bigvee_{x \in [0,s)} p(0, x)$. The metric completion $M(J)$ and the spherical completion $S(J)$ are both $[0, s] \cup \{1\}$.

Definitions 2.12. Define $w : X \to \mathcal{V}$, by $w(x) = p(x, x)$. (This is often called the *weight* function.)

Recall that a quasiuniform space $(X, \mathcal{Q})$ is *totally bounded* if the associated uniformity is totally bounded. Thus a $\mathcal{V}$-quasimetric space (X, q) is totally bounded if and only if for each $\epsilon \gg 0$ there is a finite set $F \subseteq X$ such that $X = \bigcup_{x \in F} B^s_\epsilon(x)$.

Also recall that a poset is *bounded complete* if each set that is bounded above has a supremum.

Theorem 2.13. *Let $\mathcal{V}$ be a value lattice, and let (X, p) be a $\mathcal{V}$-pmetric space.*

(a) *Let $w[X] = \{w(x) \colon x \in X\} = \{p(x, x) \colon x \in X\}$. If $(w[X], q_{p_\mathcal{V}} | w[X] \times w[X])$ is totally bounded, then each increasing net is Cauchy. Also, $(M(X), \leqslant_{\tau_p})$ and $(S(X), \leqslant_{\tau_p})$ are dcpos.*
(b) *If (X, q_p) is totally bounded, then $(w[X], q_{p_\mathcal{V}} | w[X] \times w[X])$ is also totally bounded, and $(M(X), \leqslant_{\tau_p})$ is a dcpo with compact Lawson topology.*
(c) *Let (X, q_p) be totally bounded and suppose that in $(X, \leqslant_{\tau_p})$, those pairs $x, y \in X$ that are bounded above have suprema. Then $(S(X), \leqslant_{\tau_p})$ is a bounded complete dcpo; as a result, its Lawson topology is compact.*

Proof. (a) To see that each increasing net $y = \langle y_n \rangle$, is Cauchy, note that since $w[X]$ is totally bounded, for each net $y = \langle y_n \rangle$, the net $p(y_n, y_n)$ in $\mathcal{V}$ has a Cauchy subnet, $p(y_{n_k}, y_{n_k})$. So if $\epsilon \gg 0$, find $\delta \gg 0$ such that $2\delta \leqslant \epsilon$ and let k be such that if $i, j \geqslant k$ then $|p(y_{n_i}, y_{n_i}) \dot{-} p(y_{n_j}, y_{n_j})| \leqslant \delta$. If the net is increasing and $m, p \geqslant n_k$ then for some $j \geqslant k$, $n_j \geqslant m, p \geqslant n_k$ so $q^s_\mathcal{V}(y_m, y_p) = [p(y_m, y_p) \dot{-} p(y_p, y_p)] + [p(y_m, y_p) \dot{-} p(y_m, y_m)] \leqslant 2[p(y_{n_k}, y_{n_k}) \dot{-} p(y_{n_j}, y_{n_j})] \leqslant \epsilon$, using that since p reverses order, $p(y_{n_j}, y_{n_j}) \leqslant p(y_m, y_m), p(y_p, y_p)$ and $p(y_m, y_p) \leqslant p(y_{n_k}, y_{n_k})$. This shows $\langle y_n \rangle$ is Cauchy, as required.

Thus by spherical completeness, $S(X)$ and $M(X)$ both have limits for each directed net y, and by Lemma 2.4, these limits both equal $\bigvee y[D]$, showing that directed nets have suprema, thus so do directed subsets $D \subseteq X$ (thinking of $1_D : D \to X$ in place of D).

(b) If (X, p) is totally bounded, first notice that $(w[X], q_{p_\mathcal{V}} | w[X] \times w[X])$ is as well: For let $\epsilon \gg 0$ and find $x_1, \ldots, x_n$ such that $X = \bigcup_{i=1}^n B^s_\epsilon(x_i)$. For each $y \in X$, choose a k so that $q^s_p(x_k, y) \ll \epsilon$. We assert that $p(y, y) \dot{-} p(x_k, x_k) \leqslant q^s_p(x_k, y)$: to see this note that $p(y, y) \leqslant p(x_k, x_k) + (p(x_k, y) \dot{-} p(x_k, x_k)) = p(x_k, x_k) + q_p(x_k, y)$, so $p(y, y) \dot{-} p(x_k, x_k) \leqslant q_p(x_k, y)$, and similarly, $p(x_k, x_k) \dot{-} p(y, y) \leqslant q_p(y, x_k)$. So $|p(y, y) \dot{-} p(x_k, x_k)| \leqslant q^s_p(x_k, y) \ll \epsilon$, and by the arbitrary nature of y, $w[X] \subseteq \bigcup_{k=1}^n B^s_\epsilon(p(x_k, x_k))$; since this holds for all $\epsilon \gg 0$, $w[X]$ is totally bounded, thus so is its metric completion, $M(X)$. Since $M(X)$ is also bicomplete, it is compact in the ^{s}-topology, which is the Lawson topology. Also, $M(X)$ is a dcpo by (a).

(c) By (a), $S(X)$ is a dcpo. Next we recall the common observation that each of its sets that is bounded above has a supremum: it is the supremum of the directed set of suprema of its finite subsets (which have suprema by induction on the assumption that pairs which are bounded above have suprema). The proof that each bounded complete dcpo has compact Lawson topology can be found for example, in [1]. (It uses the Alexander subbase theorem: Suppose $\mathcal{C}$ is a set of σ-closed sets, $\mathcal{F}$ is a set of $\uparrow p$'s, and $\bigcap(\mathcal{C}' \cup \mathcal{F}') \neq \emptyset$ whenever $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{F}' \subseteq \mathcal{F}$ are finite. Then whenever $\uparrow p_1, \ldots, \uparrow p_n \in \mathcal{F}$ we have that $\uparrow p_1 \cap \cdots \cap \uparrow p_n \neq \emptyset$; thus, $\{p_1, \ldots, p_n\}$ is bounded above, so it has a supremum. Thus we can set $D = \{\bigvee F \mid F \subseteq \mathcal{F} \text{ finite}\}$; D is a directed set, so it has a supremum, in P. If $C \in \mathcal{C}$, $p \in D$ then $\uparrow p \cap C \neq \emptyset$; since C is a lower set, this implies $p \in C$. Thus $D \subseteq C$, so $s = \bigvee D \in C$. Therefore $s \in (\bigcap \mathcal{C}) \cap (\bigcap \mathcal{F}) = \bigcap(\mathcal{C} \cup \mathcal{F})$, so $\bigcap(C \cup F) \neq \emptyset$.) $\square$

Example 2.14. Here is a pmetric continuous dcpo whose metric completion is not a continuous dcpo. It comes from a general construction: For each bounded metric space $M = (M, d)$ with $\phi \in M$, define $p_M : M \times M \to \mathbb{R}^+$ by

$$p_M(x, y) = \frac{d(x, y) + d(\phi, x) + d(\phi, y)}{2}.$$

Then p_M is a partial metric and ϕ is called its base point. To see (Pid), suppose $x, y \in M$. Then by the triangle inequality for d, $p_M(x, x) = 2d(\phi, x)/2 \leqslant (d(\phi, x) + d(\phi, y) + d(y, x))/2 = p_M(x, y)$.

Immediately by the symmetry of d we have the symmetry of p_M (Psy).

For (Ptr) suppose $x, y, z \in M$. Then: $p_M(x, z) = (d(x, z) + d(\phi, x) + d(\phi, z))/2 \leqslant (d(x, y) + d(\phi, x) + d(\phi, y) + d(y, z) + d(\phi, z) + d(\phi, y) - 2d(\phi, y))/2 = p_M(x, y) + (p_M(y, z) - p_M(y, y))$.

For (Pt0), suppose $x, y, z \in M$ satisfy $p_M(x,x) = p_M(x,y) = p_M(y,y)$. Then

$$\frac{d(x,x)+d(\phi,x)+d(\phi,x)}{2} = \frac{d(x,y)+d(\phi,x)+d(\phi,y)}{2}, \tag{$*$}$$

thus $d(\phi,x) = d(\phi,y) + d(x,y)$, and so $d(\phi,x) \geqslant d(\phi,y)$. Similarly, $d(\phi,y) \geqslant d(\phi,x)$, and so $d(\phi,y) = d(\phi,x)$. Thus $d(x,y) = 0$, so $x = y$.

The above shows that (M, p_M) is a partial metric space, and part of the proof of (Pt0) shows that $x \leqslant y$ (that is, $p_M(x,y) = p_M(x,x)$), if and only if $d(\phi,x) = d(\phi,y) + d(y,x)$.

Further, for our space (M, p_M), we have that $q_{p_M}(x,y) = p_M(x,y) - p_M(x,x) = (d(x,y)+d(\phi,x)+d(\phi,y))/2) - d(\phi,x) = (d(x,y) - d(\phi,x) + d(\phi,y))/2$. As a result, $q^*_{p_M}(x,y) = (d(x,y) - d(\phi,y) + d(\phi,x))/2$, and $q^s_{p_M}(x,y) = d(x,y)$.

We apply the above to the case where M is the unit disk and ϕ the origin in $\mathbb{R}^2$: $M = \{(x_1,x_2) \in \mathbb{R}^2\colon\ x^2 + y^2 \leqslant 1\}$, $\phi = (0,0)$, together with the usual Euclidean metric.

Let $L_n = \{(x_1,x_2) \in M\colon\ x_2 = \frac{1}{n}x_1,\ \frac{1}{n} \leqslant x_1\}$, $n \in \mathbb{N}$, and $L^n = \{(x_1,x_2) \in M\colon\ x_2 = \frac{1}{n}x_1,\ \frac{-1}{n} \geqslant x_1\}$, $n \in \mathbb{N}$, and define $X = (\bigcup_{n\in\mathbb{N}} L_n) \cup (\bigcup_{n\in\mathbb{N}} L^n) \cup \{\bot\}$ with $p : X \times X \to \mathbb{R}^+$ defined by: $p(x,y) = p_M(x,y)$ if $\{x,y\} \subseteq M$ and $p(x,y) = 5$ if $\{x,y\} \cap \{\bot\} \neq \emptyset$.

Several facts are now clear:

Since $q^s_{p_M} = d$, the L^n's and L_n's are ^{s}-compact (as d-continuous images of intervals).

Note that $\phi \notin X$. If $x = (x_1,x_2)$, $y = (y_1,y_2) \in X$, then by our discussion of $\leqslant$ just after the proof of (Pt0), $x \leqslant y$ if and only if y lies on a line between x and 0, so there is an $n \in \mathbb{N}$ such that $x, y \in L_n$ or $x, y \in L^n$. Thus each directed set is a subset of some L_n or some L^n, so by their ^{s}-compactness, its ^{s}-limit exists in L_n or L^n. So X is spherically complete, and we have the following facts:

(1) X is a dcpo by Lemma 2.4,
(2) p is join preserving by Theorem 2.10, and
(3) $X = S(X)$.

For each $x \in X$, x is in some L_n or some L^n, and $\Downarrow x = \{rx\colon\ rx \in M,\ r > 1\}$ if the latter set is nonempty; otherwise $\Downarrow x = \{x\}$. Therefore $\Downarrow x$ is directed with join x. Thus X is a continuous dcpo.

Now we note that $M(X) \supseteq X \cup ([-1,1] \times \{0\})$. For each $x \in [-1,0) \cup (0,1]$, $(x, \frac{1}{n}x)_{n\in\mathbb{N}}$ is a Cauchy net in X and its ^{s}-limit is $(x,0)$; also $(\frac{1}{n}, \frac{1}{n^2})_{n\in\mathbb{N}}$ is a Cauchy net in X with ^{s}-limit $(0,0)$. Since $X \subseteq X \cup ([-1,1] \times \{0\})$ and $X \cup ([-1,1] \times \{0\})$ is ^{s}-complete, $M(X) \subseteq X \cup ([-1,1] \times \{0\})$ as well.

But $M(X)$ is not a continuous dcpo; in fact, we show that $\Downarrow(0,0) = \{\bot\}$. Surely $\bot \ll (0,0)$; also $(0,0) = \bigvee[-1,0) \times \{0\}$ (so $\Downarrow(0,0) \subseteq ([-1,0) \times \{0\}) \cup \{\bot\}$); also $(0,0) = \bigvee(0,1] \times \{0\}$ (so $\Downarrow(0,0) \subseteq (0,1] \times \{0\} \cup \{\bot\}$). Thus $\Downarrow(0,0) \subseteq ((0,1] \times \{0\} \cup \{\bot\}) \cap ([-1,0) \times \{0\}) \cup \{\bot\} = \{\bot\}$. As a result, $\bigvee \Downarrow(0,0) = \bot \neq (0,0)$.

Example 2.15. Recall that a net is *left Cauchy* if for each $\epsilon \gg 0$ there is a k so that $k \leqslant m \leqslant n \Rightarrow q(x_m, x_n) \leqslant \epsilon$. A $\mathcal{V}$-quasimetric space (X, Q) is *Smyth complete* if each left Cauchy net has a τ_{q^s}-limit. Clearly each Cauchy net is left Cauchy, so each Smyth complete space is ^{s}-complete. As a result, any Smyth complete space Y, containing X also contains $M(X)$, and so in general (and in particular, in Example 2.14) $Y \neq S(X)$. So Example 2.14 is also an example of a continuous dcpo which is not Smyth complete.

3. Spherical completion and round ideal completion

Definition 3.1. Given a poset (partially ordered set) $(P, \leqslant)$, an *auxiliary relation* on $(P, \leqslant)$, is a binary relation $\prec$ such that if $p, q, r, s \in P$, then:

(Str) $q \prec p$ implies $q \leqslant p$ ($\prec$ is at least as strict as $\leqslant$);
(Tra) $s \leqslant q$, $q \prec p$ and $p \leqslant r$ implies $s \prec r$ ($\prec$ is transitive through $\leqslant$);
(Int) if $r \prec p$, then $r \prec s \prec p$ for some s (interpolation); and
(Sbd) if $q, r \prec p$, then there is some t such that $q, r \leqslant t$ and $t \prec p$ (subdirected—the set of elements preceding any p is directed).

Note that (Tra) in the presence of the other axioms, implies the usual transitivity for $\prec$, since if $p \prec q$ and $q \prec r$ then $p \leqslant q$ and $q \prec r$, so $p \prec r$.

For auxiliary relations we use the notations $\Uparrow p = \{x\colon\ p \prec x\}$ and $\Downarrow p = \{x\colon\ x \prec p\}$. The axiom (Sbd) says that for each p, $\Downarrow p$ is directed (by $\leqslant$); but in fact, $\Downarrow p$ is directed by $\prec$: for if $q, r \in \Downarrow p$ then $q, r \prec p$, so by (Sbd) for some t, $q, r \leqslant t$ and $t \prec p$, so by (Int) for some s, $t \prec s$ and $s \prec p$; thus $s \in \Downarrow p$ and by (Tra), $q, r \prec s$. An auxiliary relation $\prec$ is *approximating* if for each $p \in P$, $p = \bigvee \Downarrow p$. More can be learned about auxiliary relations in [1] and [3].

It is helpful to write some of these axioms in terms of set inclusions:

$(Str)\quad \Leftrightarrow \quad \twoheaddownarrow q \subseteq \downarrow q,$

$(Tra)\quad \Leftrightarrow \quad q \prec p$ implies $\uparrow p \subseteq \twoheaduparrow q$ and $\downarrow q \subseteq \twoheaddownarrow p$.

We now consider the associated bitopological space and round ideal completion of a poset with an approximating auxiliary relation:

Definitions 3.2. Given a poset with approximating auxiliary relation, $(P, \leqslant, \prec)$:

Its *pseudoScott topology*, ρ is the topology generated by $\{\twoheaduparrow p\colon\ p \in P\}$, and

a subset $I \subseteq P$ is a *round ideal* if I is a lower set directed by $\prec$. The *round ideal completion* of $(P, \leqslant, \prec)$, $R(P)$, is its set of round ideals with the partial order $\subseteq$, and the embedding map $j : P \to R(P)$ defined by $j(a) = \twoheaddownarrow a$.

For any continuous poset $(P, \leqslant)$, its way-below relation, $\ll$ is an approximating auxiliary relation, and $\sigma = \rho$ (see [1]). The round ideal completion is discussed and further referenced in [5, pp. 242–249].

The following result is well known; we refer to [8, 2.3], where it is given in essentially our notation: If $(P, \leqslant, \prec)$ is a poset with approximating auxiliary relation, then $(R(P), \subseteq)$ is a continuous dcpo and the map $j : P \to R(P)$ is:

$(*)$ a bitopological imbedding of (P, ρ, ω) into $(R(P), \sigma, \omega)$, such that
$(*)$ for any $p, q \in P$, $p \leqslant q \Leftrightarrow j(p) \subseteq j(q)$, and $p \prec q \Leftrightarrow j(p) \ll j(q)$.

Further, if all finite sets in P that are bounded above have suprema, and all $\prec$-directed sets have upper bounds, then $R(P)$ is bounded complete.

Given a set X, a *Urysohn relation on* X is a relation on the power set, $(2^X, \subseteq)$ that satisfies (Str), (Tra) and (Int). Given two sets X, Y with Urysohn relations, $\lhd_X, \lhd_Y$ on them, a *Urysohn map* is an $f : X \to Y$ such that whenever $A \lhd_Y B$ then $f^{-1}[A] \lhd_X f^{-1}[B]$.

Thus Urysohn relations would be a special case of auxiliary relations but for lack of the axiom (Sbd), which is dropped, but can be put in a larger relation $\lhd'$ since 2^X has sups of pairs, by saying $p \lhd' q$ if for some finite set $\{p_1, \dots, p_n\}$, each $p_i \lhd q$ and $p \leqslant \bigvee_{k=1}^n p_k$. But this weakening of axioms is convenient for topology in important ways:

- Each Urysohn relation $\lhd$ has a *dual*, $\lhd^*$, defined by $A \lhd^* B$ if $X \setminus B \lhd X \setminus A$. It is easy to see that $\lhd^*$ satisfies (Str), (Tra) and (Int), so it is a Urysohn relation. But it does not always satisfy (Sbd) even if $\lhd$ does. Note that $\lhd = (\lhd^*)^*$.

Another obvious way to take care of the issue of (Sbd) and duality is to define a *quasiproximity* to be a Urysohn relation $\lhd$ such that both $\lhd$ and $\lhd^*$ both satisfy (Sbd). Equivalently: for each finite set F:

(Sbd) if $B \lhd A$ for each $B \in F$, then $\bigcup F \lhd A$, and
(Sbd*) if $A \lhd B$ for each $B \in F$, then $A \lhd \bigcap F$.

Using $A = \emptyset$, $F = \emptyset$ in (Sbd) yields $\emptyset \lhd \emptyset$, and letting $A = X$, $F = \emptyset$ in (Sbd*) yields $X \lhd X$. For any Urysohn relation $\lhd$, there is a smallest quasiproximity $\lhd^P$ containing $\lhd$, which is found by closing $\lhd$ under the operations indicated in (Sbd) and (Sbd*).

- Each Urysohn relation $\lhd$, gives rise to a topology, $\tau_\lhd$, by taking as open sets those $T \subseteq X$ such that whenever $x \in T$ then for some finite number of sets, $B_1, \dots, B_n \subseteq X$, $\{x\} \lhd B_k$ for each k and $\bigcap_{k=1}^n B_k \subseteq T$. In this topology, $\{x\} \lhd B$ if and only if B is a neighborhood of x. Also, if $(X, \lhd_X)$ and $(Y, \lhd_Y)$ are sets with Urysohn relations, each Urysohn map $f : X \to Y$ is continuous from (X, τ_X) to (Y, τ_Y). As a result, each gives rise a bitopological space, $(X, \tau_\lhd, \tau_{\lhd^*})$, and given two sets with Urysohn relations, $(X, \lhd_X)$ and $(Y, \lhd_Y)$, each Urysohn map $f : X \to Y$ is pairwise continuous from $(X, \tau_{\lhd_X}, \tau_{\lhd_X^*})$ to $(Y, \tau_{\lhd_Y}, \tau_{\lhd_Y^*})$.
- Urysohn's lemma holds for sets with Urysohn relations. In particular, a slight variation of the usual proof of Urysohn's lemma shows that whenever $A \lhd B$, there is a Urysohn map $f : (X, \lhd) \to (\mathbb{I}, \lhd_<)$ such that $f[A] = \{1\}$ and $f[X \setminus B] = \{0\}$.
- Quasiuniformities, thus quasimetrics and partial metrics, give rise to quasiproximities. If (X, q) is a $\mathcal{V}$-quasimetric space, then for $A, B \subseteq X$, $A \lhd_q B$ if for some $r \gg 0$, $N_r(A) \subseteq B$, where $N_r(A) = \{y\colon$ for some $x \in A,\ q(x, y) \leqslant r\}$. To see that (Int) holds for $\lhd_q$, let $N_r(A) \subseteq B$, $r \gg 0$; then $2s \leqslant r$ for some $s \gg 0$, so letting $C = N_s(A)$, we have $N_s(C) \subseteq N_r(A)$ by the triangle inequality, so $A \lhd_q C \lhd_q B$. For (Sbd), if $B \lhd A$ for each $B \in F$, then for each such B let $N_{r_B}(B) \subseteq A$, and note that for any $s \gg 0$, if $s \leqslant r_B$ then for each $B \in F$, $N_s(\bigcup F) \subseteq \bigcup_{B \in F} N_{r_B}(B) \subseteq A$, so $\bigcup F \lhd A$, and similarly if $A \lhd B$ for each $B \in F$, then let $N_{r_B}(A) \subseteq B$ for each $B \in F$ and note that $N_s(A) \subseteq \bigcap_{B \in F} N_{r_B}(A) \subseteq \bigcap F$, so $A \lhd \bigcap F$; again (Str) and (Tra) are left to the reader. Construction of the quasiproximity associated with a quasiuniformity is similar, and found in [4]. Of course for partial metrics we define $\lhd_p = \lhd_{q_p}$.

- In [4] it is pointed out that each quasiproximity is associated with a unique totally bounded quasiuniformity is $\mathcal{Q}_{\lhd}$. This quasiuniformity is generated by sets of the form $(A \times B) \cup ((X \setminus A) \times X)$, and of course its ^{s}-completion is ^{s}-compact and pairwise completely regular. For any totally bounded quasiuniformity $\mathcal{Q}$, the bicompletion $(\bar{X}, \bar{\mathcal{Q}})$, $\tau_{\bar{\mathcal{Q}}} \vee \tau_{\bar{\mathcal{Q}}^*}$ is compact, and $(\bar{X}, \tau_{\bar{\mathcal{Q}}}, \tau_{\bar{\mathcal{Q}}^*})$ is pairwise completely regular (see [4] or [7], following 2.6).

A Urysohn relation is T_0 if for each $x, y \in X$ such that $x \neq y$, either $\{x\} \lhd X \setminus \{y\}$ or $\{y\} \lhd X \setminus \{x\}$. Clearly this holds if and only if $\tau_{\lhd}$ is T_0.

Theorem 3.3. *Given a set X with T_0 quasiproximity $\lhd$, there is a totally bounded partial metric p into the value lattice $\mathcal{V} = \mathbb{I}^K$, such that $\lhd = \lhd_p$.*

Proof. Let K denote the set of all Urysohn maps $f : X \to \mathbb{I}$, and let $p : X \times X \to \mathbb{I}^K$ be defined by $p(x, y)(f) = f(x) \vee f(y)$. Then each $f \in K$ is uniformly continuous with respect to $\mathcal{Q}_{\lhd}$, thus it extends uniquely to a uniformly continuous on the ^{s}-completion, $\bar{f} : \bar{X} \to \mathbb{I}$. Since $(\bar{X}, \tau_{\bar{\mathcal{Q}}_{\lhd}}, \tau_{\bar{\mathcal{Q}}^*_{\lhd}})$ is ^{s}-compact, each pairwise continuous function is uniformly continuous. Thus $f \to \bar{f}$ is a one-one onto map from K to $\bar{K}$, the set of pairwise continuous maps from $\bar{X}$ to $\mathbb{I}$.

By the discussion preceding this theorem, $(\bar{X}, \tau_{\bar{\mathcal{Q}}_{\lhd}}, \tau_{\bar{\mathcal{Q}}^*_{\lhd}})$ is also pairwise completely regular, so it arises from the $\mathcal{V}$-pmetric, $\bar{p} : \bar{X} \times \bar{X} \to \mathbb{I}^K$, defined by $\bar{p}(x, y)(\bar{f}) = \bar{f}(x) \vee \bar{f}(y)$ (the proof is straightforward and given in [9, 2.9]). By ^{s}-compactness, there is only one quasiuniformity giving rise to this space (see [4]), thus $\mathcal{Q}_{\bar{p}} = \bar{\mathcal{Q}}_{\lhd}$; also this quasiuniformity is totally bounded, thus so is $\bar{p}$. Clearly $p = \bar{p}|X \times X$, so p is a totally bounded partial metric on X and $\mathcal{Q}_p = \mathcal{Q}_{\lhd}$, therefore $\lhd_p = \lhd$. $\square$

Auxiliary relations also give rise to Urysohn relations: If $\prec$ is an auxiliary relation on $(P, \leqslant)$ then for $A, B \subseteq P$, $A \lhd_{\prec} B$ if for some $p, q \in P$, $A \subseteq \uparrow q \subseteq \Uparrow p \subseteq B$. The interpolation property (Int) of $\lhd_{\prec}$ results from that of $\prec$: if $A \subseteq \uparrow q \subseteq \Uparrow p \subseteq B$ then $q \in \Uparrow p$ so $p \prec q$ thus for some r, $p \prec r \prec q$. Letting $C = \uparrow r$, we have $A \subseteq \uparrow q \subseteq \Uparrow r \subseteq C$ and $C = \uparrow r \subseteq \Uparrow p \subseteq B$, so $A \lhd_{\prec} C \lhd_{\prec} B$. Proof of (Str) and (Tra) are left to the reader.

In [3, 1.4], it is recalled that if $\prec$ is an auxiliary relation, then the topology $\tau_{\lhd_{\prec}}$ arising from $\lhd_{\prec}$, is ρ. Also, the topology $\tau_{\lhd^*_{\prec}}$ is the lower topology ω.

For any continuous dcpo, such as $R(P)$, $\ll$ is an auxiliary relation (see [1] or [5]), thus $\sigma = \tau_{\lhd_{\ll}} = \rho$ and $\tau_{\lhd^*_{\ll}} = \omega$. Thus for some set K, $(R(P), \sigma, \omega)$ arises from an $\mathbb{I}^K$-pmetric $p : R(P) \times R(P) \to \mathbb{I}^K$ ([9], proof of 2.9). Note that $\mathbb{I}$ is compact, so $\mathbb{I}^K$ is as well; therefore $\mathbb{I}^K$ is totally bounded. In particular, $w[P] \subseteq \mathbb{I}^K$ is totally bounded (that ^{s}-compact spaces and subspaces of totally bounded spaces are totally bounded are textbook results; see for example, [6, Chapter 6, Theorem 32]). Further, since $\sigma = \tau_p$, p is join preserving by Theorem 2.10.

Theorem 3.4.

(a) *Let $(P, \leqslant)$ be a poset with an approximating auxiliary relation $\prec$. Whenever p is a totally bounded partial metric into a value lattice $\mathcal{V}$, and $\lhd_{\prec} = \lhd_p$, then the spherical completion $S(P)$ is the round ideal completion $R(P)$.*
(b) *Further, if pairs $x, y \in P$ that are bounded above have suprema, then $S(P) = M(P)$ as well and the induced bitopological space, $(R(P), \sigma, \omega)$ is joincompact.*

Proof. (a) Since $S(P)$ is the smallest subspace of the bicompletion of $(P, \mathcal{Q}_p) = (P, \mathcal{Q}_{\lhd_{\prec}})$ that is closed under directed suprema, it will do to show that $R(P)$ is also this space, which we do by showing that it also has the universal mapping property: each Urysohn map $f : (P, \lhd_{\prec}) \to (Y, \lhd_q)$, (Y, q), a spherically complete quasimetric space extends uniquely to $R(P)$. Since any such map is pairwise continuous from $(R(P), \sigma, \omega)$ to (Y, τ_q, τ_{q^*}), it is continuous with respect to their Hausdorff joins, that is, from $(R(P), \lambda)$ to (Y, τ_{q^s}), so it is uniquely determined by f.

Now we show that each Urysohn map $f : (P, \lhd_{\prec}) \to (Y, \lhd_q)$, has such an extension. Note that for $I \in R(P)$, I is directed and since as a continuous map, f is order-preserving, $f[I]$ is also directed; also since p is totally bounded, $f[I]$ is Cauchy as well by Theorem 2.13, so $\lim^s f[I] = \bigvee f[I] \in Y$. Consider $\hat{f} = \{(I, \bigvee f[I]) \colon I \in R(P)\}$, this is, as required, a uniformly continuous map that extends f by the usual argument (given for example in the proof of Theorem 2.5(b)).

(b) For each net $x = (x_n)_{n \in D}$ consider $I_x = \{a \in P \colon$ for some $z \in P$, $n \in D$, $a \prec z \leqslant x_m$ whenever $n \leqslant m\}$. To see that $I_x \in R(P)$ note first that if $a \leqslant b \in I_x$ then for some $m \in D$, $y \in P, m \leqslant k \Rightarrow b \prec_p z \leqslant x_k$, so $a \prec_p z \leqslant x_k$, thus $a \in I_x$, showing that T_x is a lower set. Next, if $a, b \in I_x$, then for some $m, n \in D$, $y, z \in P, m \leqslant k \Rightarrow a \prec_p z \leqslant x_k$ and $n \leqslant k \Rightarrow b \prec_p y \leqslant x_k$. There is thus a $j \in D$ such that $m, n \leqslant j$, and so if $j \leqslant k$ then $a \prec_p z \leqslant x_k$ and $b \prec_p y \leqslant x_k$. Therefore the pair $\{y, z\}$ is bounded above, so it has a supremum, $w = y \vee z$. By the above and (Tra), $a, b \prec_p w$; by (Sbd) for some c, $a, b \prec_p c \prec_p w \leqslant x_k$ whenever $j \leqslant k$; thus $a, b \prec c \in I_x$. Therefore I_x is a round ideal.

We now suppose that x is Cauchy, and show that $\lim^s(x) = \bigvee(I_x)$. Of course if $a \in I_x$ then for some $z \in P$, $n \in D$, $z \leqslant x_m$ if $n \leqslant m$, so $a \leqslant \lim^s(x)$; this shows that $\bigvee(I_x) \leqslant \lim^s(x)$. For the reverse inequality, first notice that if $z \in P$ and for some $n \in D$, $z \leqslant x_m$ if $n \leqslant m$, then $a \prec z \Rightarrow a \in I_x$, thus $z = \bigvee(\Downarrow z) \leqslant \bigvee(I_x)$. Now let $\epsilon \gg 0$; then choose $\delta \gg 0$ such that $2\delta \leqslant \epsilon$ and $n \in D$ so that $|x_m \dot{-} x_p| \leqslant \delta$ whenever $n \leqslant m, p$ and note that if $n \leqslant m$ then we have $x_n \dot{-} x_m \leqslant \delta$, so $x_n \dot{-} \delta \leqslant x_m$, thus by the previous sentence, $x_n \dot{-} \delta \leqslant \bigvee(I_x)$ and therefore $x_n \leqslant \bigvee(I_x) + \delta$. But these inequalities also say that whenever $m \geqslant n$ then $x_m \in \downarrow(x_n + \delta)$, an ^{s}-closed set, so $\lim^s(x) \leqslant x_n + \delta \leqslant \bigvee(I_x) + \epsilon$. By the arbitrary nature of $\epsilon \gg 0$, we have $\lim^s(x) \leqslant \bigvee(I_x) + \bigwedge\{\epsilon \gg 0\} = \bigvee(I_x)$, since $(\mathcal{V}, \geqslant)$ is a value lattice. □

Example 3.5. The following space, closely related to the one discussed in Example 2.14, shows that the assumption that $\lhd_{\prec} = \lhd_p$ for our totally bounded partial metric cannot be weakened to the bitopological assumption that $\tau_{\lhd_{\prec}}$ is ρ and $\tau_{\lhd^*_{\prec}}$ is ω. It is a partial metric space (Y, p) such that $(Y, \leqslant_p)$ is a continuous poset, but $S(Y)$ is not a continuous dcpo.

Let $Y = [-1, 0) \cup (0, 1] \times \{0\}$, considered as a subspace of the space (M, p_M) discussed in Example 2.14. Then:

$$\Downarrow(x, 0) = \begin{cases} (x, 1] \times \{0\}, & 0 < x < 1, \\ [-1, x) \times \{0\}, & -1 < x < 0, \\ \{(x, 0)\}, & x = \pm 1. \end{cases}$$

As a result each $(x, 0) = \bigvee \Downarrow(x, 0)$, showing that $\gg$ is an approximating auxiliary relation for the poset $(Y, \leqslant_p)$. But $S(Y) = M(Y) = [-1, 1] \times \{0\}$, and as in Example 2.14, $\Downarrow(0, 0) = \emptyset$, so $\bigvee \Downarrow(0, 0) \neq (0, 0)$.

The reader may check that p_M gives rise to the bitopological space, (Y, σ, ω). But it does not yield the quasiproximity $\lhd_{\prec}$, since for each positive r,

$$N_r\big((0, 1] \times \{0\}\big) \cap [-1, 0) \times \{0\} \neq \emptyset.$$

4. Open issues

The term “spherically complete” comes from the theory of valued fields, where it is used to generalize Hensel’s Lemma. A generalized metric space is called *spherically complete* if decreasing intersections of closed balls are nonempty. Our use of this term assumes that we are thinking about the space of formal balls, rather than the original space. But these ideas are not developed here.

Further, note that proper consideration of proximity was centrally important in the above. We believe that it captures the essence of domain theory more than do those of bitopology or quasiuniformity, and we hope to emphasize this in future work.

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